GRAPH THEORY INTRODUCTION

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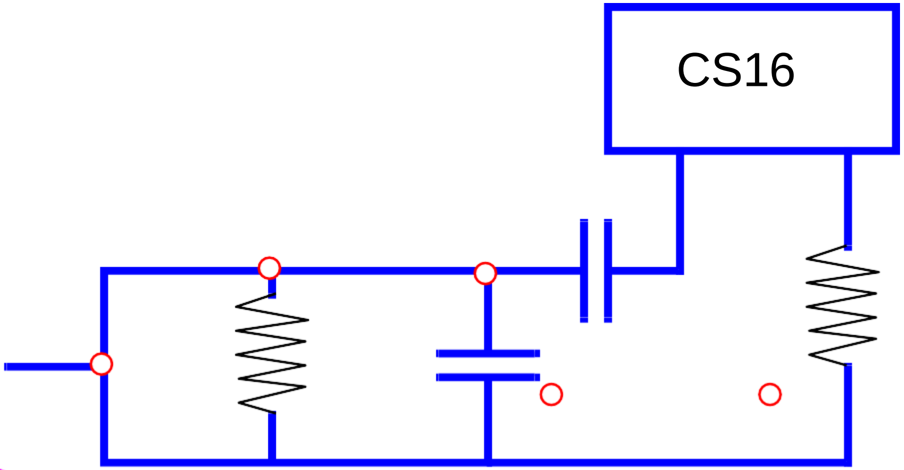
Basic Graph Definitions

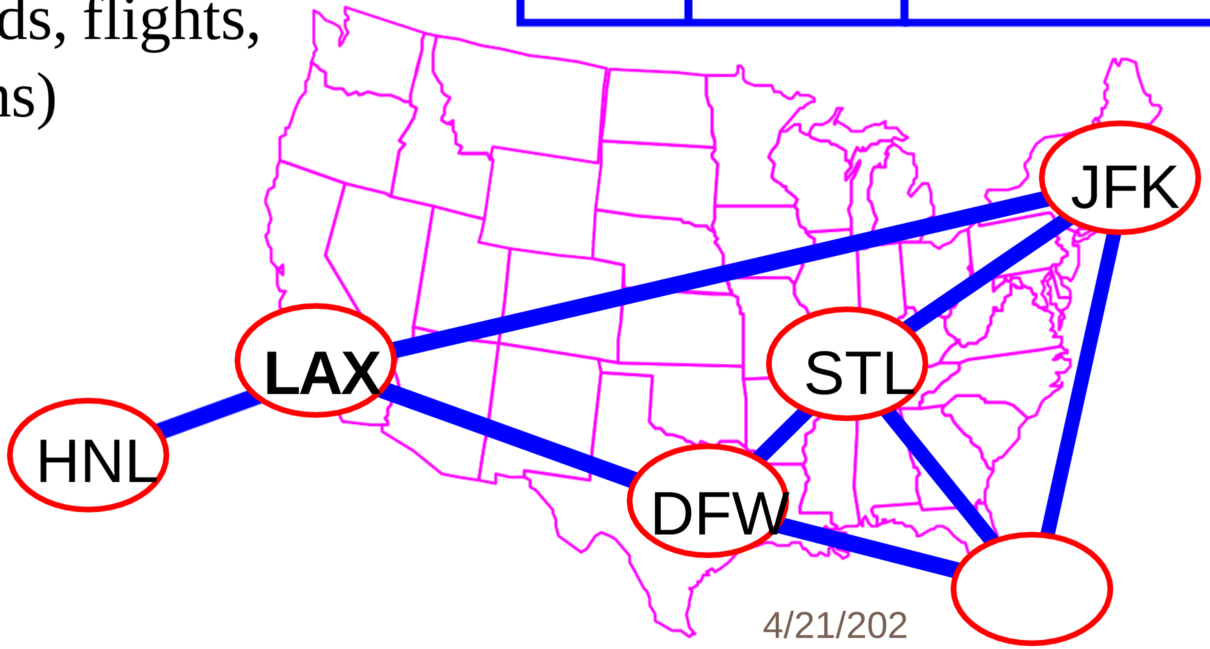
- A data structure that consists of a set of nodes (*vertices*) and a set of edges that relate the nodes to each other
- The set of edges describes relationships among the vertices. Some Examples,
  - Car navigation system
  - Efficient database
  - Build a bot to retrieve info of WWW
  - Representing computational models

Applications

- electronic circuits



- networks (roads, flights, communications)



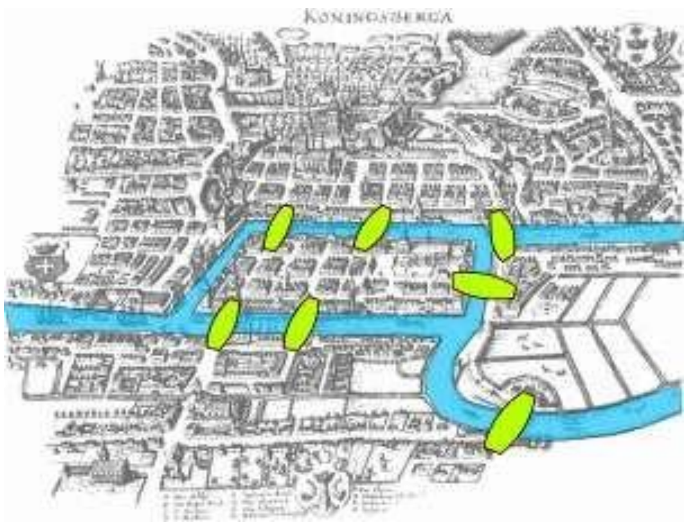
Classic Graph Theory Problems

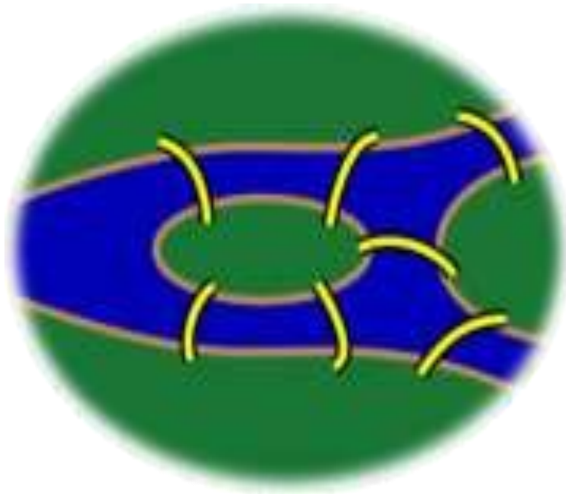
Graph theory started from a mathematical curiosity.

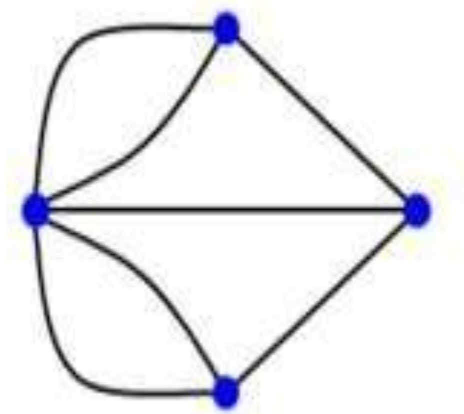
"The Seven Bridges of Königsberg is a problem inspired by an actual place and situation. The city of Kaliningrad, Russia (at the time, Königsberg, Germany) is set on the Pregolya River, and included two large islands which were connected to each other and the mainland by seven bridges. The question is whether it is possible to walk with a route that crosses each bridge exactly once, and return to the starting point. In 1736, Leonhard Euler proved that it was not possible."

Seven Bridges of Königsberg

"In proving the result, Euler formulated the problem in terms of graph theory, by abstracting the case of Königsberg -- first, by eliminating all features except the landmasses and the bridges connecting them; second, by replacing each landmass with a dot, called a vertex or node, and each bridge with a line, called an edge or link. The resulting mathematical structure is called a graph."







Seven Bridges of Königsberg

"The shape of a graph may be distorted in any way without changing the graph itself, so long as the links between nodes are unchanged. It does not matter whether the links are straight or curved, or whether one node is to the left of another.

Euler realized that the problem could be solved in terms of the degrees of the nodes. The degree of a node is the number of edges touching it; in the Königsberg bridge graph, three nodes have degree 3 and one has degree 5. Euler proved that a circuit of the desired form is possible if and only if there are no nodes of odd degree. Such a walk is called an ***Eulerian circuit or an Euler tour.*** *Since the* ***graph corresponding to Königsberg has four nodes of odd degree, it cannot have an Euler's Circuit."***

Euler"s Theory

<u>Euler path:</u>

- *A graph is said to be containing an Euler path if it can be traced in 1 sweep without lifting the pencil from the paper and without tracing the same edge more than once. Vertices may be passed through more than once. The starting and ending points need not be the same.*

<u>Euler circuit:</u>

- *An Euler circuit is similar to an Euler path, except that the starting and ending points must be the same.*

Euler"s Theory

Graph	Number of odd vertices (vertices connected to an odd number of edges)	Number of even vertices (vertices connected to an even number of edges)	What does the path contain? (Euler path = P; Euler circuit = C; Neither = N)
1	0	10	C
2	0	6	C
3	2	6	P
4	2	4	P
5	4	1	N
6	8	0	N

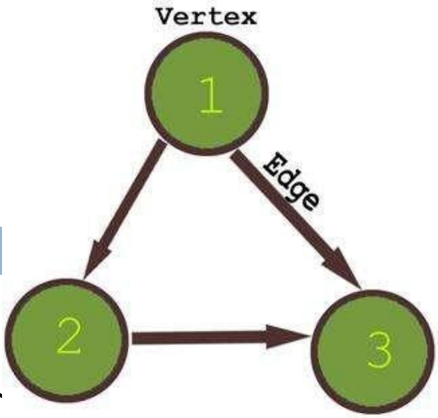
Euler"s Theory

- From the above table, we can observe that:
  - A graph with all vertices being even contains an Euler circuit.
  - A graph with 2 odd vertices and some even vertices contains an Euler path.
  - A graph with more than 2 odd vertices does not contain any Euler path or circuit.

Seven Bridges of Königsberg

"The problem can be modified to ask for a path that traverses all bridges but does not have the same starting and ending point. Such a walk is called an Eulerian trail or Euler walk. Such a path exists if and only if the graph has exactly two nodes of odd degree, those nodes being the starting and ending points. (So this too was impossible for the seven bridges of Königsberg.)"

Formal Definition:

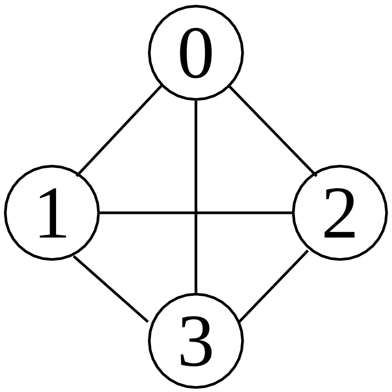


- A graph, *G=(V, E)*, consists of two sets:
  - a finite non empty set of ***vertices(V)***, and
  - a finite set ***(E)*** *of unordered pairs of distinct vertices called edges.*
  - *V(G)* and *E(G)* represent the sets of vertices and edges of *G*, respectively.
- Vertex: In graph theory, a **vertex** (plural **vertices**) or **node** or points is the fundamental unit out of which graphs are formed.
- Edge or Arcs or Links: Gives the relationship between the Two vertices.

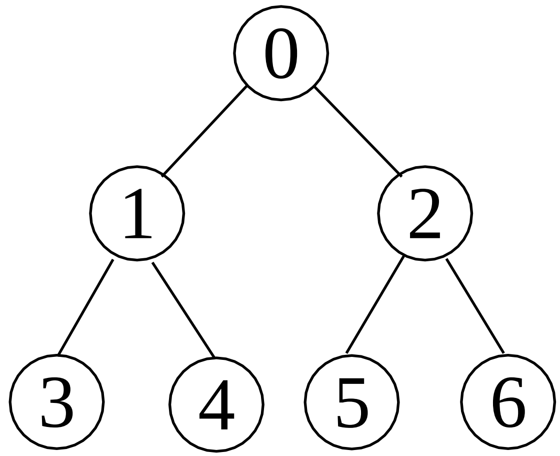
Graph Terminology

- Two vertices joined by an edge are called the end vertices or endpoints of the edge.
- If an edge is directed its first endpoint is called the origin and the other is called the destination.
- Two vertices are said to be adjacent if they are endpoints of the same edge.

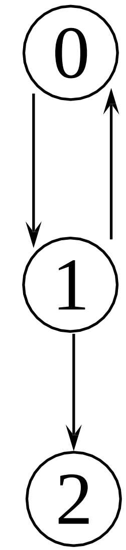
Examples for Graph



G_1



G_2



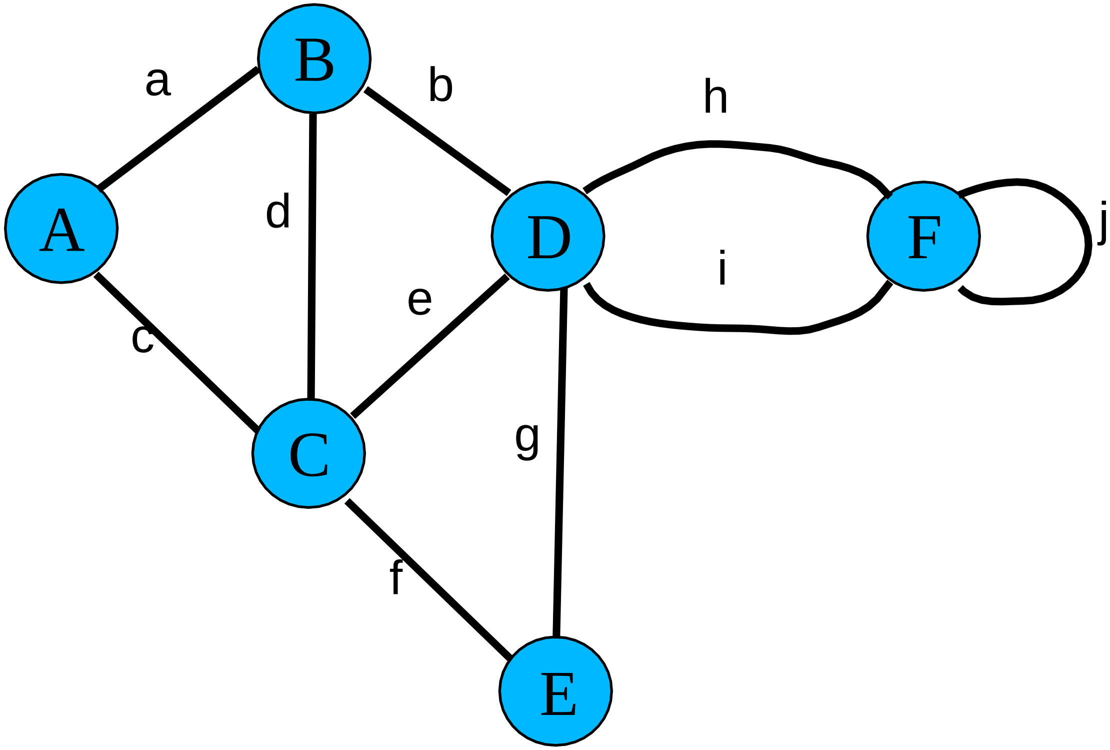
G_3

$V(G_1)=\{0,1,2,3\}$ $E(G_1)=\{(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}$

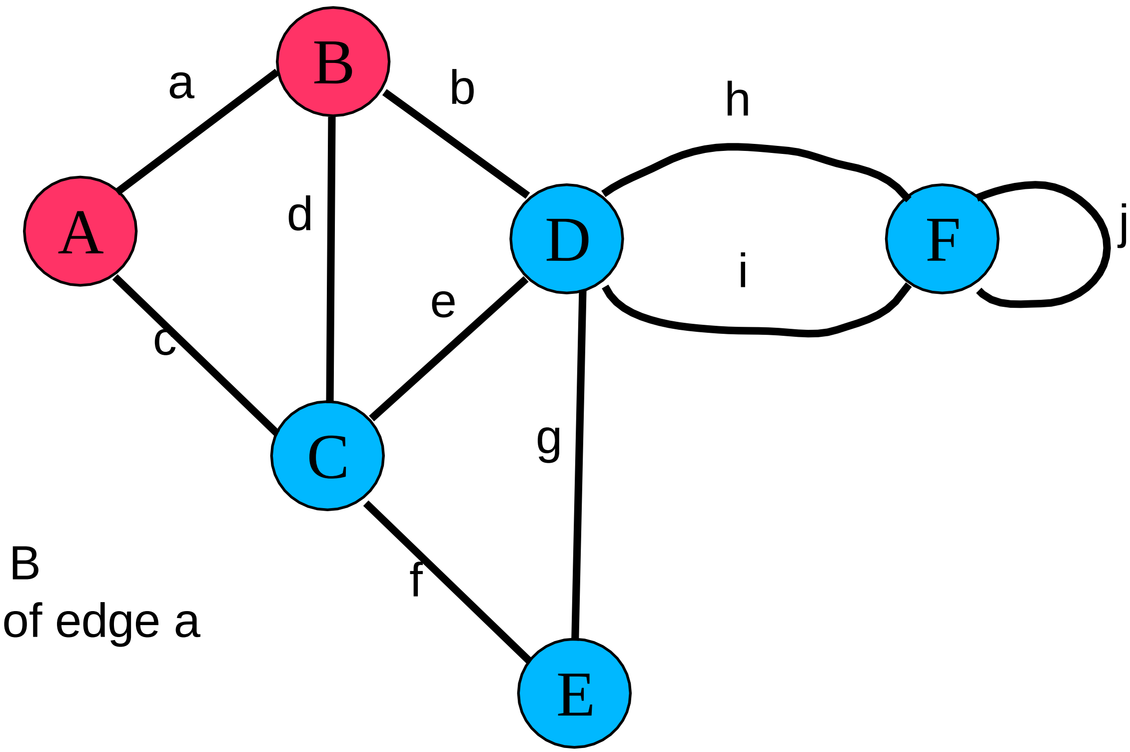
$V(G_2)=\{0,1,2,3,4,5,6\}$ $E(G_2)=\{(0,1),(0,2),(1,3),(1,4),(2,5),(2,6)\}$

$V(G_3)=\{0,1,2\}$ $E(G_3)=\{<0,1>,<1,0>,<1,2>\}$

Graph Terminology

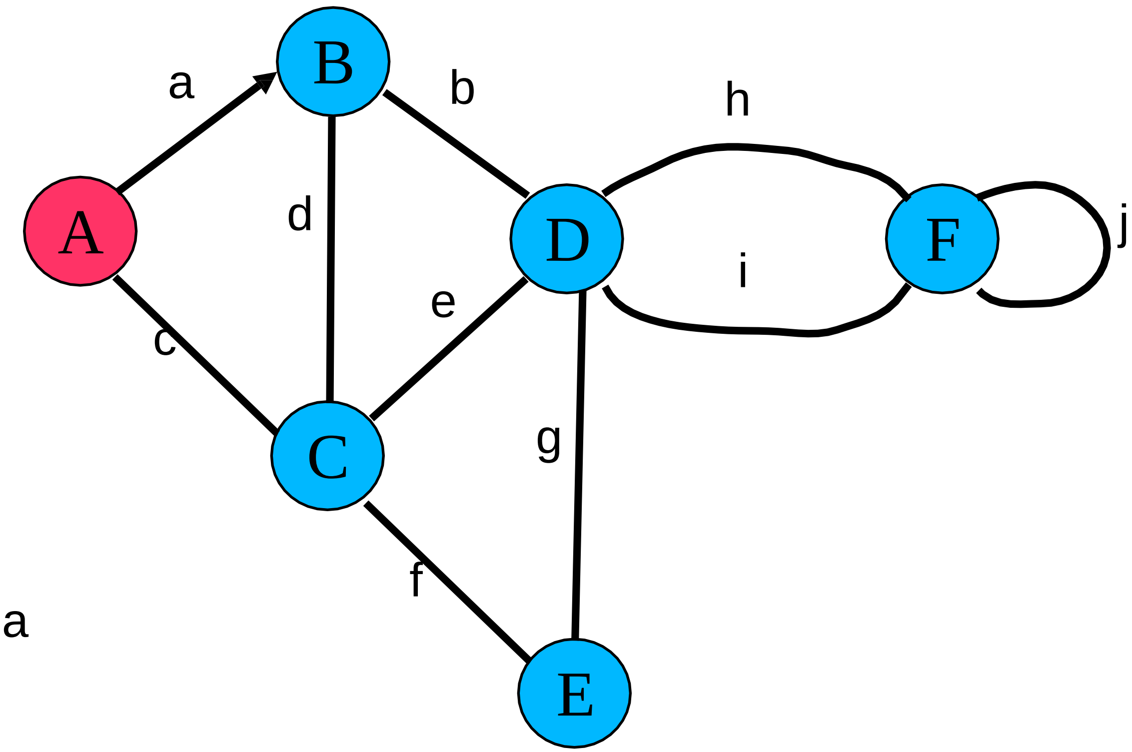


Graph Terminology



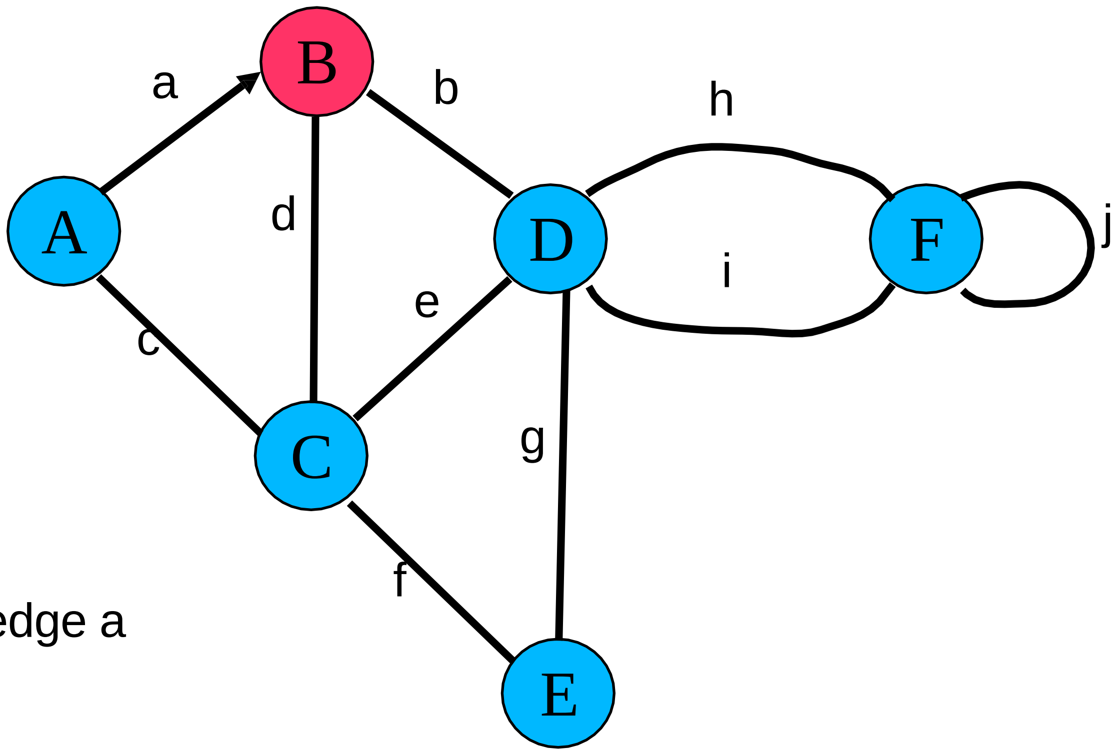
Vertices A and B
are endpoints of edge a

Graph Terminology



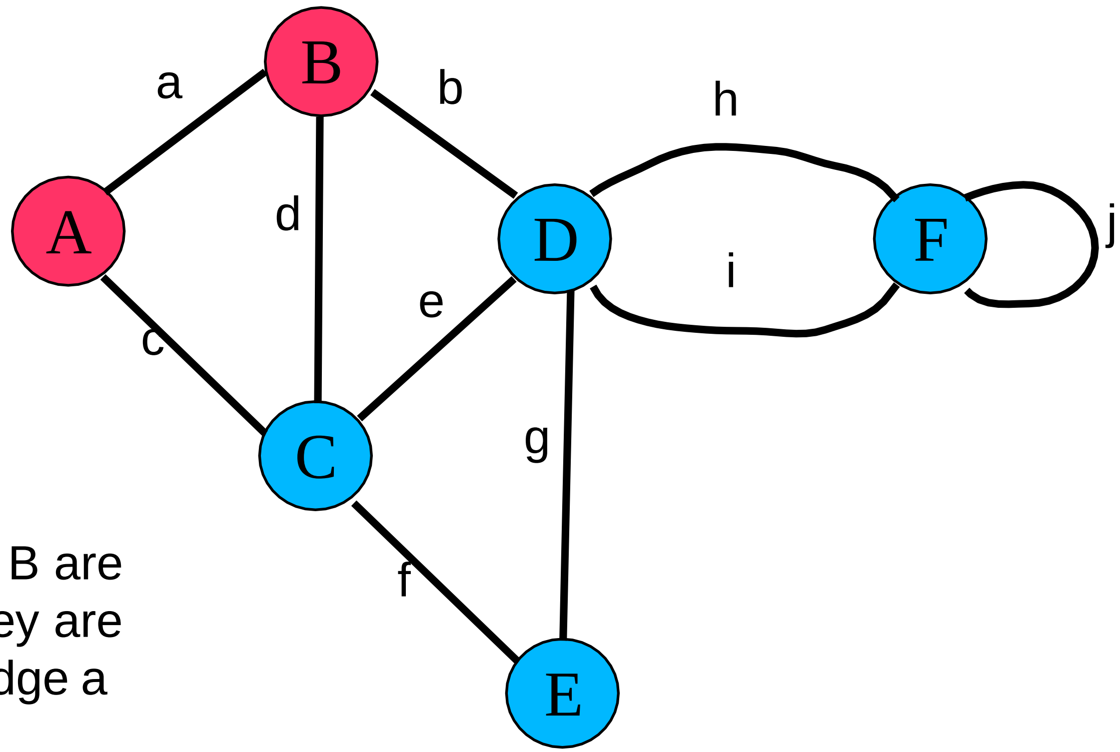
Vertex A is the origin of edge a

Graph Terminology



Vertex B is the destination of edge a

Graph Terminology



Vertices A and B are adjacent as they are endpoints of edge a

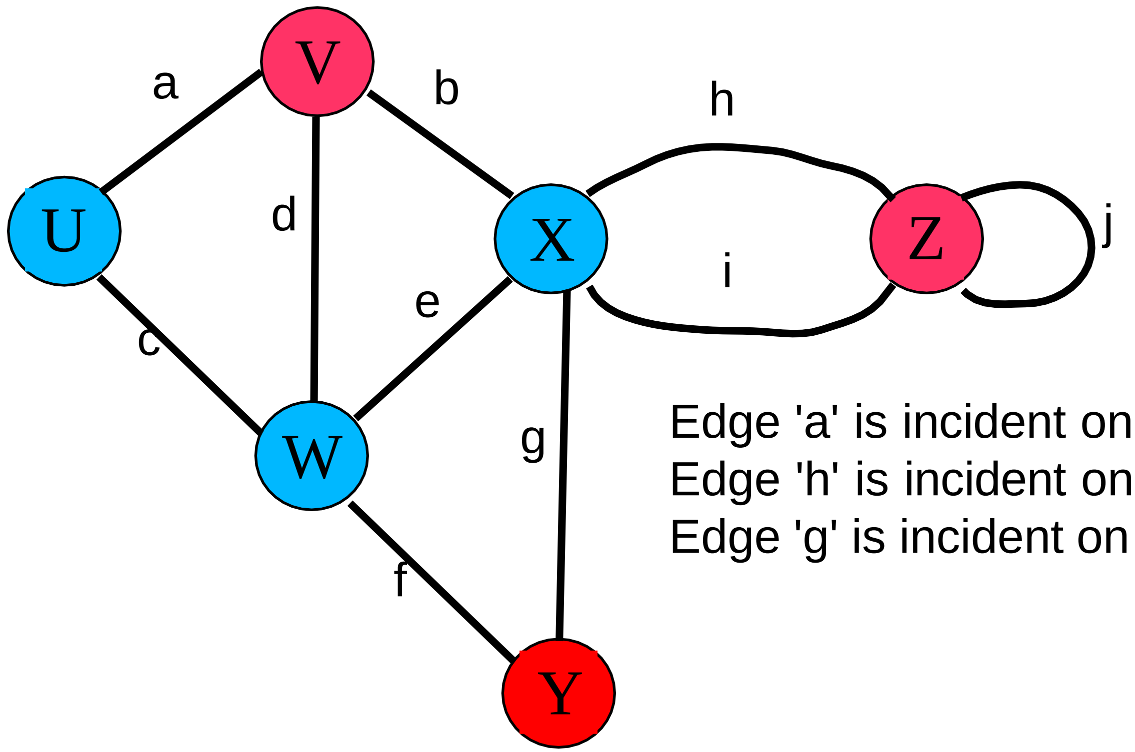
Graph Terminology

- An edge is said to be incident on a vertex if the vertex is one of the edges endpoints.

- The outgoing edges of a vertex are the directed edges whose origin is that vertex.

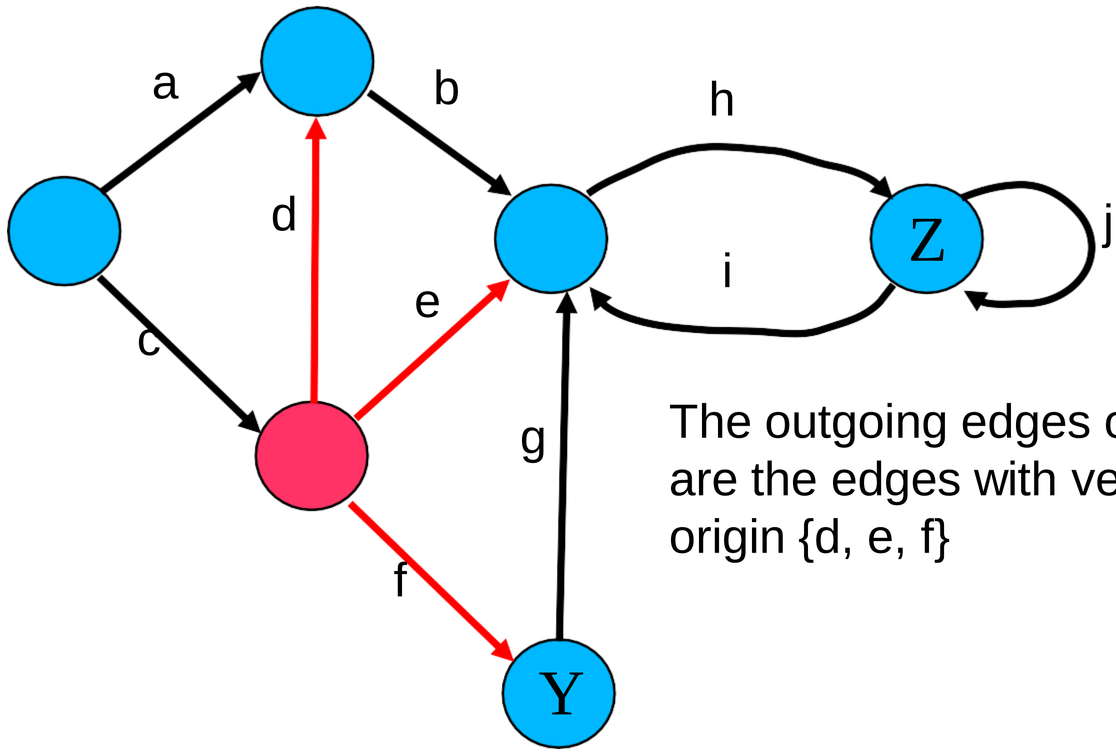
- The incoming edges of a vertex are the directed edges whose destination is that vertex.

Graph Terminology



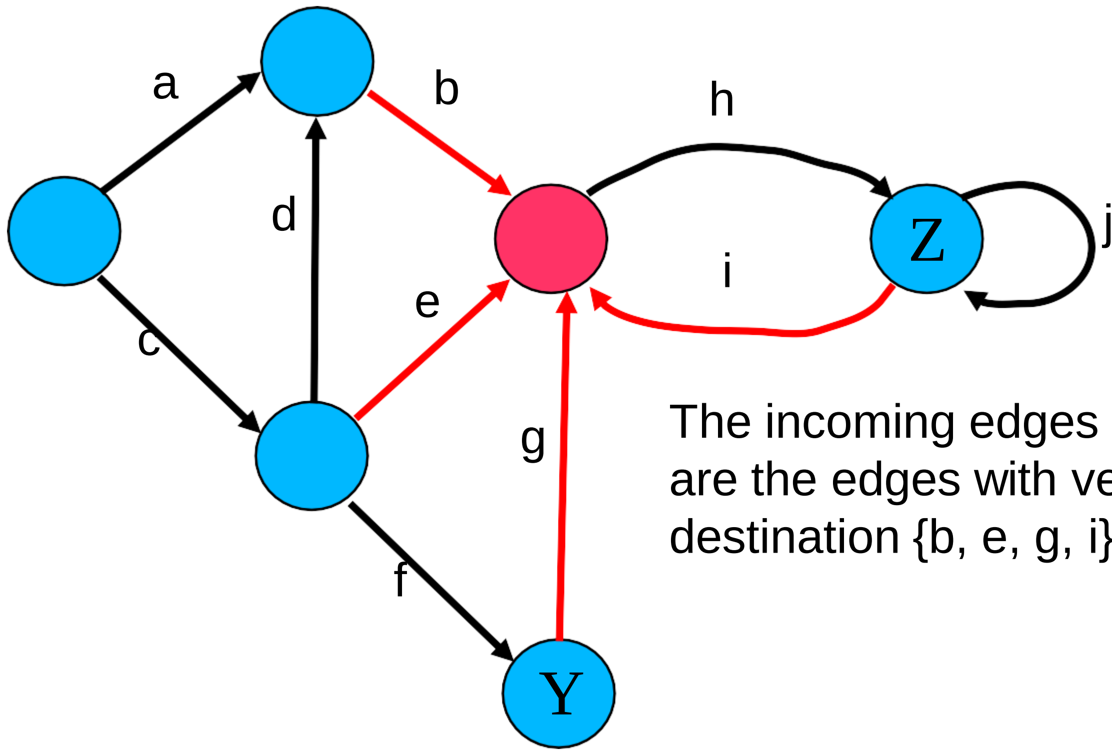
Edge 'a' is incident on vertex V
Edge 'h' is incident on vertex Z
Edge 'g' is incident on vertex Y

Graph Terminology



The outgoing edges of vertex W are the edges with vertex W as origin {d, e, f}

Graph Terminology

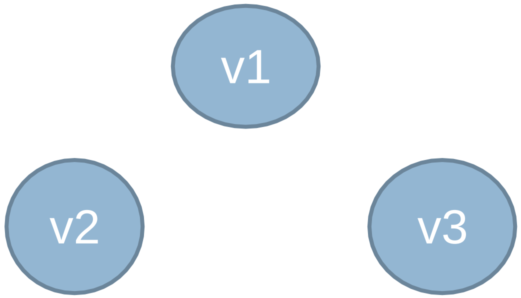


The incoming edges of vertex X are the edges with vertex X as destination {b, e, g, i}

Types of Graph

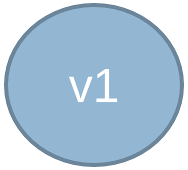
Null graph, Trivial Graph

- A graph G=(V,E) where E=0 is said to be Null or Empty graph



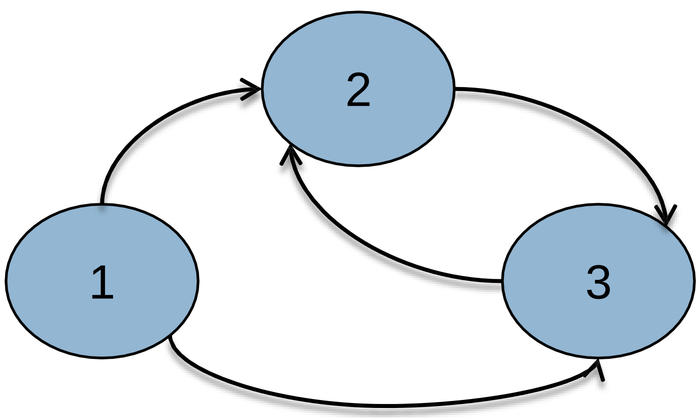
- A graph with One vertex

and no edge is called as a trivial graph.



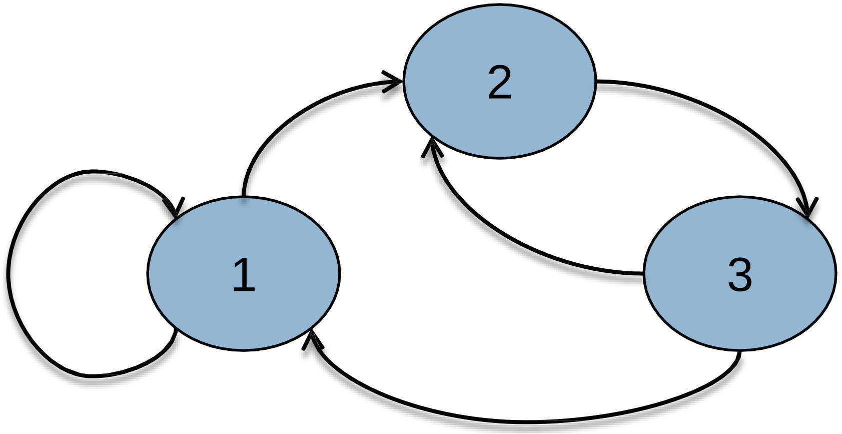
MultiGraph(Without Self Edge)

- The term multigraph refers to a graph in which multiple edges between vertices are permitted.
- A multigraph G = (V, E) is a graph which has the set of vetrices and multiple edges between vertices.



MultiGraph(With Self Edge)

- A multidigraph is a directed graph which is permitted to have *multiple edges,* i.e., edges with the own,source and target vertices.



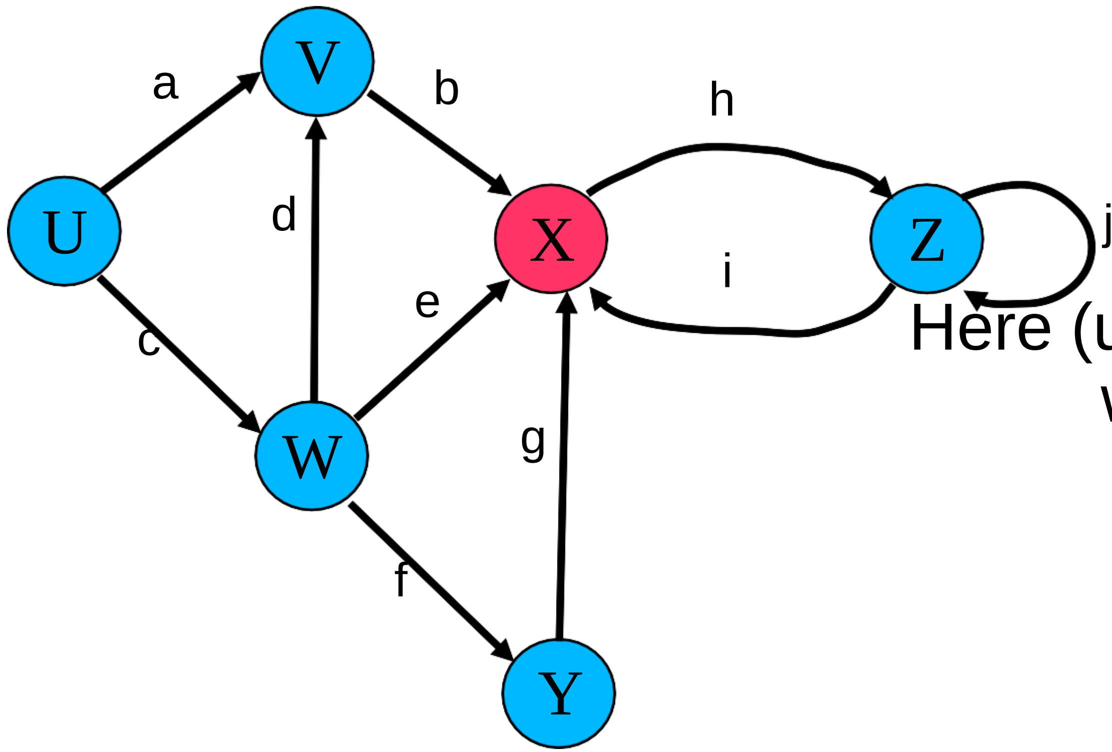
Directed Graph

- A directed graph is one in which every edge (u, v) has a direction, so that (u, v) is different from (v, u)

There are two possible situations that can arise in a directed graph between vertices u and v.

- i) only one of (u, v) and (v, u) is present.
- ii) both (u, v) and (v, u) are present.

Directed Graph



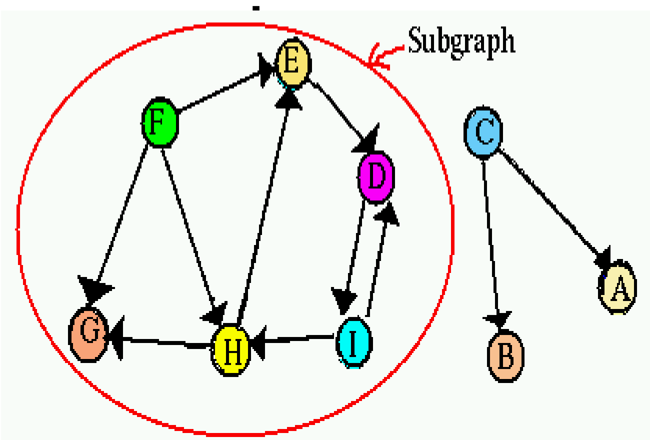
Here (u,v) is possible where as (v,u) is not possible.

In a directed edge, u is said to be adjacent to v and v is said to be adjacent from u.

The edge <u,v> is incident to both u and v.

Directed Graph

- Directed Graphs are also called as **Digraph.**
- Directed graph or the digraph make reference to edges which are directed (i.e) edges which are Ordered pairs of vertices.
- The edge(uv) is referred to as <u,v> which is distinct from <v,u> where u,v are distinct



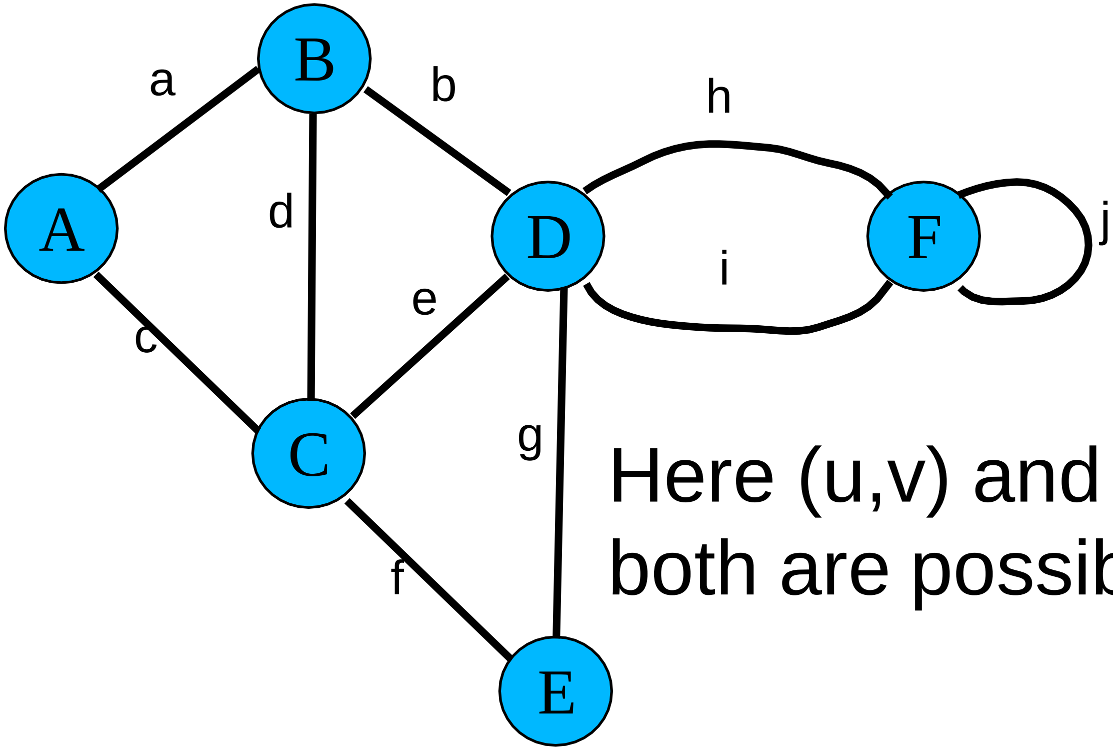
Undirected Graph

- In an undirected graph, there is no distinction between (u, v) and (v, u).
- An edge (u, v) is said to be directed from u to v if the pair (u, v) is ordered with u preceding v.

  E.g. A Flight Route
- An edge (u, v) is said to be undirected if the pair (u, v) is not ordered

  E.g. Road Map

Undirected Graph



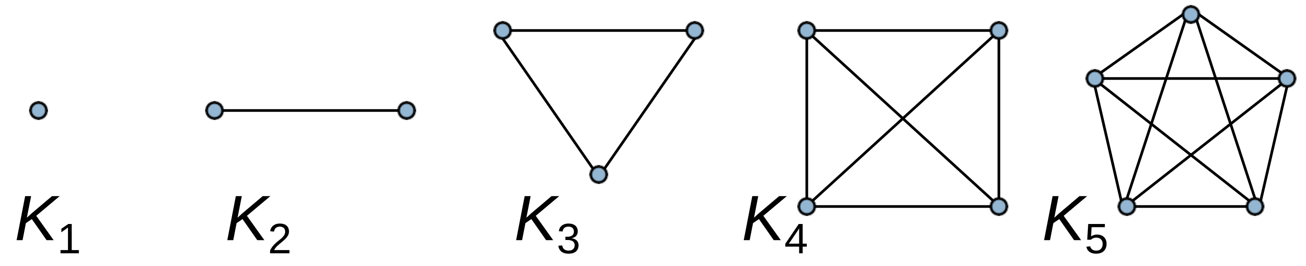
Here (u,v) and (v,u) both are possible.

Undirected Graph

- A graph whose definition makes reference to Unordered pairs of vertices as Edges is known as undirected graph.
- Thus an undirected edge (u,v) is equivalent to (v,u) where u and v are distinct vertices.
- In the case of undirected edge(u,v) in a graph, the vertices u,v are said to be adjacent or the edge(u,v) is said to be incident on vertices u,v.

Complete Graph

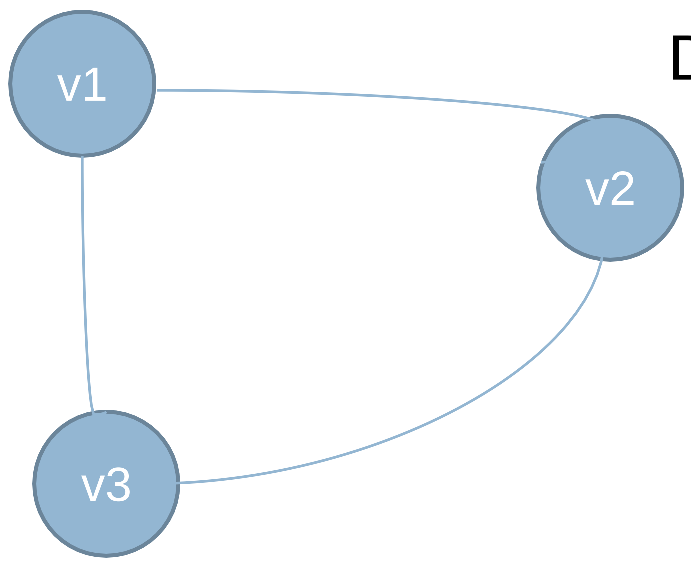
- In a complete graph: Every node should be connected to all other nodes.
- The above means " Every node is adjacent to all other nodes in that graph".
- The degree of all the vertices must be same.



- K_n = Denotes a complete with n number of vertices.

Complete Undirected Graph

An undirected graph with „n" number of vertices is said to be complete ,iff each vertex



Number of vertices=3

Degree of Each vertices

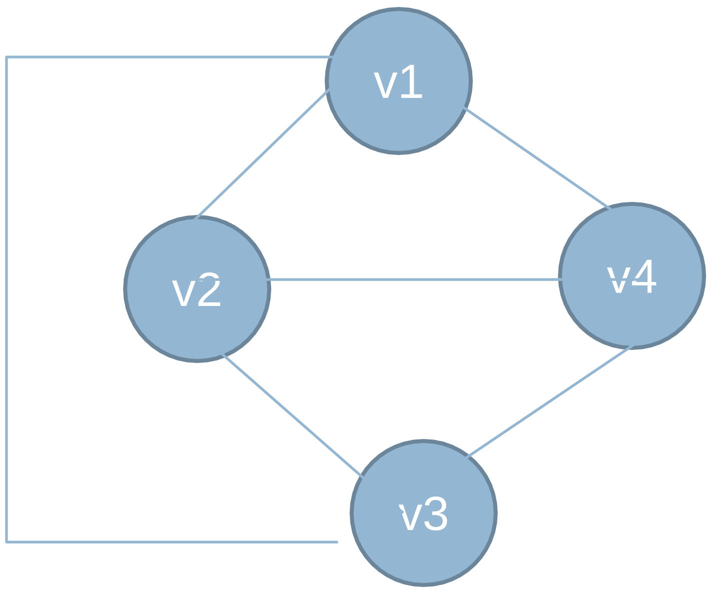
=(n-1)

=(3-1)

=2

Complete Undirected Graph

- An n vertex undirected graph with exactly (n.(n-1))/2 edges is said to be complete.



Here we have 4 number of vertices and hence

(4.(4-1))/2= (4.3)/2

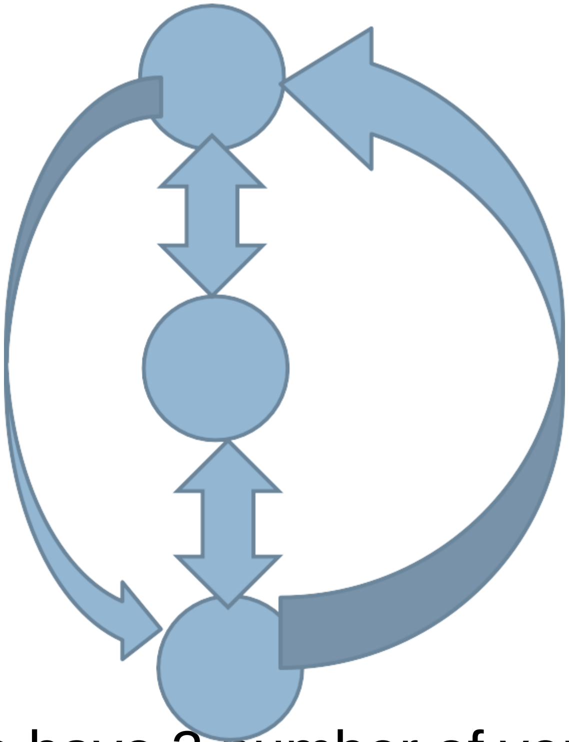
=6

Hence the graph has 6 number of edges and it is a Complete Undirected graph.

Complete Directed Graph

- An directed graph with „n‟ number of vertices is said to be complete ,if each vertices has (n-1) number of in-coming and out-going edges.
  - In case of a digraph with n vertices, maximum number of edges is given by n.(n-1).Such a graph with exactly n.(n-1) edges is said to be Complete digraph.

Complete Directed Graph Example:

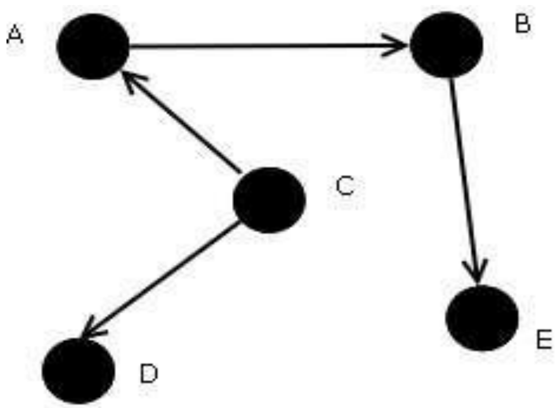


Hence the graph has 6 number of edges and it is a Complete directed graph.

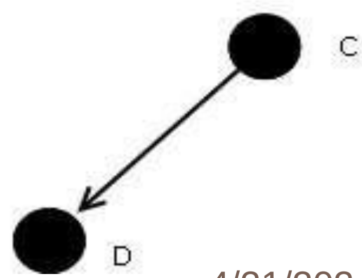
- Here we have 3 number of vertices and hence
  - n.(n-1)= 3.(3-1)
  - =6

Sub Graph

- A graph whose vertices and edges are subsets of another graph.
- A subgraph G‟=(V‟,E‟) of a graph G = (V,E) such that V‟ ⊆ V and E‟ ⊆ E , Then G is a supergraph for G‟.

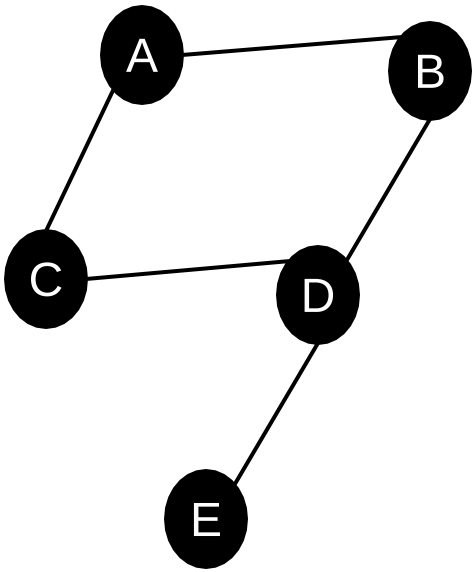


(G)

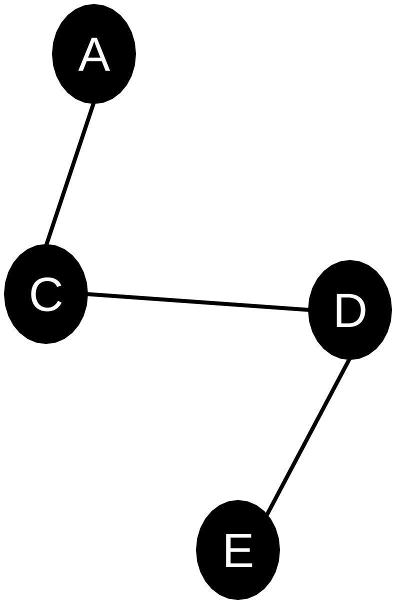


(G1)

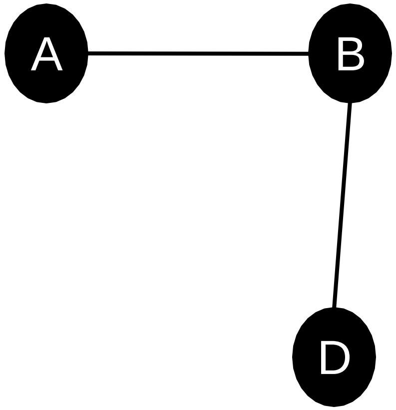
..Sub Graph



(G)



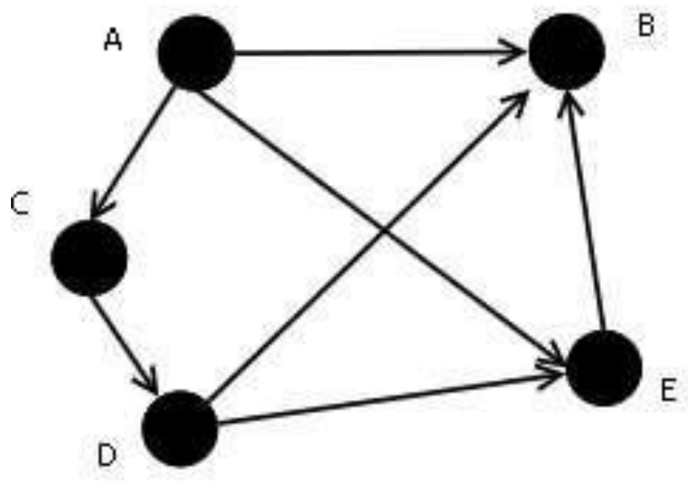
(G1)

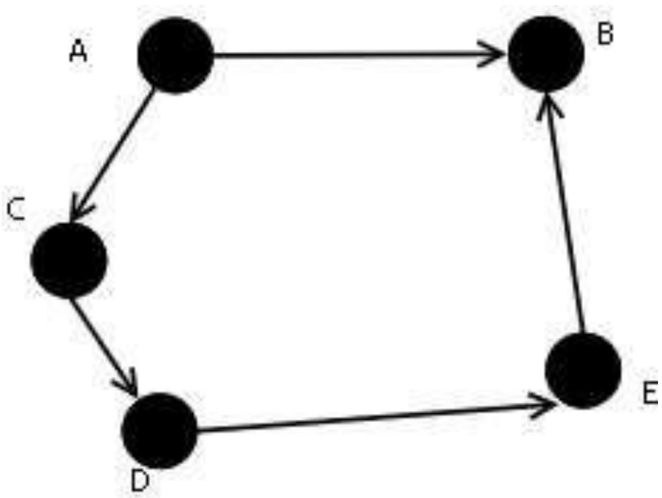


(G2)

Spanning Subgraph

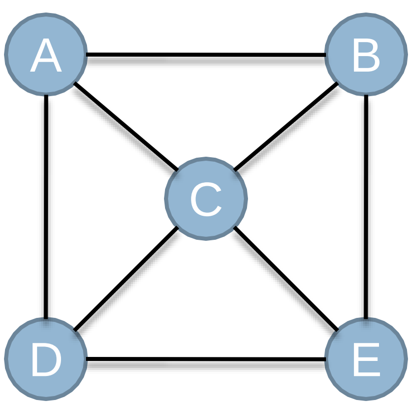
- A *spanning subgraph* is a subgraph that contains all the vertices of the original graph.

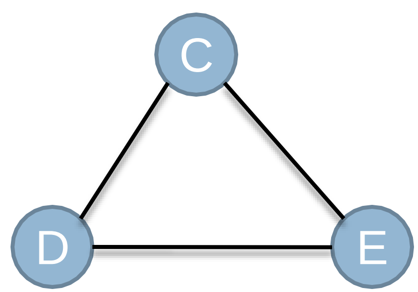


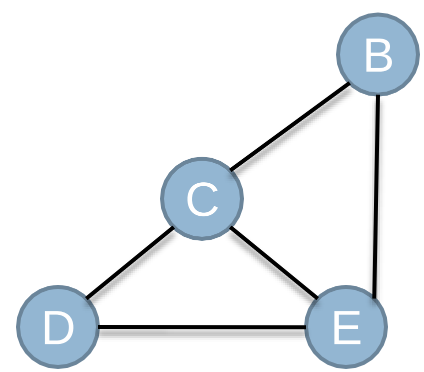


Induced-Subgraph

- Vertex-Induced Subgraph:
  - A *vertex-induced subgraph* is one that consists of some of the vertices of the original graph and all of the edges that connect them in the original.

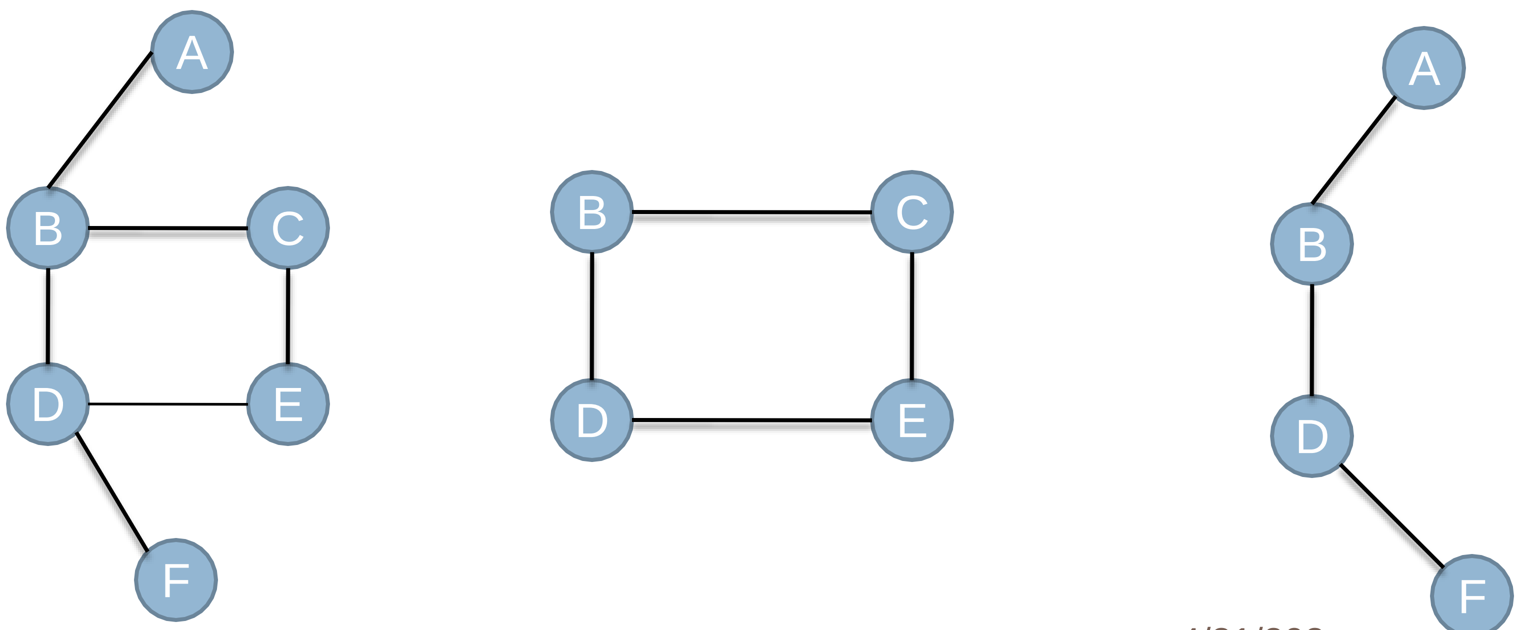






Induced-Subgraph

- Edge-Induced Subgraph:
  - An *edge-induced subgraph* consists of some of the edges of the original graph and the vertices that are at their endpoints.



Conclusions

Graph theory enables us to study and model networks and solve some difficult problems inherently capable of being modelled using networks.

Various terms e.g. vertex and edge, are associated with graph theory which gives these terms special meanings. These meanings need to be understood and remembered in order to apply graph theoretic approaches to solving problems.

When solving a problem by developing a graph-based program, careful attention must be given at the design stage to the structuring of data to help make solving the problem tractable, to enable linkages to be traced efficiently and to avoid duplication of data.